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Centres of Convex Sets in L^p Metrics

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It is shown that for each convex body $A \subset \mathbf{R}^n$ there exists a naturally defined family $\mathcal{G}_A \subset \mathbf{C}(S^{n-1})$ such that for every $g \in \mathcal{G}_A$, and every convex function $f: \mathbf{R} \to \mathbf{R}$ the mapping $y \mapsto \int_{S^{n-1}} f(g(x) - \langle y, x \rangle)\, d\sigma(x)$ has a minimizer which belongs to A. As an application, approximation of convex bodies by balls with respect to L^p metrics is discussed.

1. Introduction

Let $\mathcal{K}^n$ be the family of all convex, compact and non-empty subsets of $\mathbf{R}^n$. As is well known, the Hausdorff distance between two members of $\mathcal{K}^n$ can be described as the L^∞ distance between the restrictions to the unit sphere of their support functions. In the same way one can define other L^p metrics. These metrics have been discussed in various contexts by McClure and Vitale [9, 17], Saint-Pierre [12], and Florian [5]. One can consider the following problem: Given $A \in \mathcal{K}^n$, what can be said about single-point sets which are the best approximations to A in these L^p metrics? One can deduce immediately from the fact that L^p spaces are uniformly convex, whenever $1 < p < \infty$, that there exists a unique element $m_p(A)$ of $\mathbf{R}^n$ such that its singleton is the solution to the above problem. Now, a more detailed question arises: What can be said about the location of $m_p(A)$ relative to A? One of our goals is to prove that actually $m_p(A)$ is an element of A. This result will follow from our Theorem 3.2 which seems to be of some independent interest. For further information on the topic of approximation of convex bodies the reader is referred to Gruber's surveys [7, 8].

In the sequel, every mapping from $\mathcal{K}^n$ into $\mathbf{R}^n$ which is a *selection* will be called a *centre*. Three kinds of centres are widely known and have numerous applications; these are the *baricentre*, the *Steiner point* (see: [6, 10, 11], and especially [12] which is an excellent monograph on this

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subject) and the *Chebyshev centre*. We define the baricentre of $A \in \mathcal{K}^n$ by the formula

$$b(A) = (1/m(A)) \int_A x \, dm(x),$$

where m is the Lebesgue measure on the minimal flat containing A. Recall that the Chebyshev centre of A is the centre of the smallest ball which contains A (Note that it is unique as the underlying norm is the Euclidean norm.) It was observed by Saint-Pierre [12, Sect. 4] that the Steiner point of A coincides with the point $m_2(A)$. In turn, the Chebyshev centre of A is equal to the point $m_\infty(A)$. As we shall see in Section 4, these centres play an important role in certain asymptotic formulae. A formula of that kind appears in [1] and [12]. It is shown that if B^n is the unit ball centered at the origin in $\mathbf{R}^n$ then $\lim_{t \to \infty} b(A + tB^n) = m_2(A)$. Let us notice that this result is an immediate consequence of the Steiner formula for the so-called *quermassvectors* (see [14]).

2. Preliminaries

A function $\varphi: \mathbf{R}^n \to \mathbf{R}$ is *subadditive* if for any $x, y \in \mathbf{R}^n$,

$$\varphi(x + y) \leqslant \varphi(x) + \varphi(y).$$

It is *positively homogeneous* if for every $x \in \mathbf{R}^n$ and every $\alpha \in [0, +\infty)$,

$$\varphi(\alpha x) = \alpha \varphi(x).$$

The *support function* h_A of $A \in \mathcal{K}^n$ is defined by the formula

$$h_A(x) = \sup\{\langle a, x \rangle : a \in A\},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product. The restriction of h_A to the unit sphere is denoted by $\bar{h}_A$.

Obviously, support functions are subadditive and positively homogeneous. As is well known, and easily deducible from the Hahn–Banach theorem, the converse is also true.

Suppose that γ is a function defined on some subset M of $\mathbf{R}^n$. We shall say that the *growth of γ is majorized by a support function φ on M* if for any $x, y \in M$,

$$\gamma(x) - \gamma(y) \leqslant \varphi(x - y).$$

LEMMA 2.1. *For $M \subset \mathbf{R}^n$, let $\gamma: M \to \mathbf{R}$ be a mapping whose growth is majorized by a support function h_A on M. Then*

1. *there exists an extension of γ to the whole $\mathbf{R}^n$ whose growth is majorized by h_A on $\mathbf{R}^n$; such an extension is given by the formula*

$$\tilde{\gamma}(x) = \sup\{\gamma(m) - h_A(m-x) \colon m \in M\};$$

2. *$\tilde{\gamma}$ is differentiable almost everywhere and the gradient $\nabla\tilde{\gamma}(x)$ belongs to A, whenever it exists.*

Proof. 1. We shall follow the classical proof of the Hahn–Banach theorem. The majorized growth of γ and the subadditivity of h_A imply that

$$\gamma(z) - \gamma(y) \leqslant h_A(z-x) + h_A(x-y)$$

or equivalently

$$\gamma(z) - h_A(z-x) \leqslant \gamma(y) + h_A(x-y).$$

The latter inequality assures us that $\tilde{\gamma}$ is well defined. Let us fix two points $u, v \in \mathbf{R}^n$ and $\varepsilon > 0$. By the definition of $\tilde{\gamma}$, there exists $m \in M$ such that

$$\tilde{\gamma}(u) - \varepsilon < \gamma(m) - h_A(m-u).$$

Hence

$$\tilde{\gamma}(u) - \tilde{\gamma}(v) < \gamma(m) - h_A(m-u) + \varepsilon - (\gamma(m) - h_A(m-v)) \leqslant h_A(u-v) + \varepsilon.$$

2. Since $\tilde{\gamma}$ has its growth majorized by a support function, it must be Lipschitz continuous. By Rademacher's theorem (see e.g. [13] for a simple proof), $\tilde{\gamma}$ is differentiable almost everywhere. Suppose that x is a point of differentiability of $\tilde{\gamma}$, and observe that by the homogeneity of h_A,

$$\frac{\tilde{\gamma}(x+ty) - \tilde{\gamma}(x)}{t} \leqslant h_A(y),$$

whenever $y \in \mathbf{R}^n$ and $t > 0$. It now is easily seen that letting t tend to 0 one obtains

$$\langle \nabla\tilde{\gamma}(x), y \rangle \leqslant h_A(y). \quad \blacksquare$$

Given a set X and a function $f: X \to \mathbf{R}$. The set of all *minimizers* of f will be denoted by $\operatorname{Min} f$, that is, $\operatorname{Min} f = \{y : f(y) \leqslant f(x)$, for every $x \in X\}$. If f has exactly one minimizer then it is denoted by $\min f$.

PROPOSITION 2.2. *Suppose that there are given a directed set Λ, a compact topological space X and a net of continuous functions $f_\lambda\colon X \to \mathbf{R}$, $\lambda \in \Lambda$. If f_λ, $\lambda \in \Lambda$, converges uniformly to $f\colon X \to \mathbf{R}$ then $\operatorname{Min} f_\lambda$ converges to $\operatorname{Min} f$ in the upper topology, that is, for every open set $V \subset X$ which contains $\operatorname{Min} f$ there exists λ_0 such that $\operatorname{Min} f_\lambda \subset V$, whenever $\lambda > \lambda_0$.*

3. A LOCATION THEOREM

In order to prove Theorem 3.2, we will need the following version of the Gauss–Green formula.

PROPOSITION 3.1. *Let S^{n-1} be the unit sphere in $\mathbf{R}^n$, σ the Lebesgue measure on S^{n-1}, and dx the infinitesimal element of the n-dimensional volume. Suppose that $g\colon B^n \to \mathbf{R}$ is a Lipschitz continuous function. Then its gradient ∇g is defined almost everywhere. Moreover, ∇g is Lebesgue measurable and essentially bounded on B^n, and*

$$\int_{S^{n-1}} g(x)\, x \, d\sigma(x) = \int_{B^n} \nabla g(x)\, dx.$$

Remarks 3.1. 1. Proposition 3.1 slightly differs from those versions of the Gauss–Green formula which are usually reproduced in textbooks, for there is most often assumed that the function g is at least continuously differentiable. However, because of the simplicity of the domain of integration considered here, the ordinary proof of the Gauss–Green formula (see e.g. [15, Ch. VI, §7] or [2, ch. 5]) still works with rather mild modifications. On the other hand, there exists a very general theorem due to Federer (see [3] or [4]) which implies our proposition.

2. In the context of centres, formula of that kind has been used in [11] and [16].

THEOREM 3.2. *Suppose that there are given a convex function $f\colon \mathbf{R} \to \mathbf{R}$, a set $A \in \mathcal{K}^n$, a function $\gamma\colon S^{n-1} \to \mathbf{R}$ whose growth is majorized by h_A and a function $F\colon \mathbf{R}^n \to \mathbf{R}$ defined by the formula*

$$F(y) = \int_{S^{n-1}} f(\gamma(x) - \langle y, x \rangle)\, d\sigma(x). \qquad (*)$$

Then $\operatorname{Min} F \cap A \neq \emptyset$. A fortiori, if f is strictly convex then F has a unique minimizer which belongs to A.

Proof. The theorem will be proved in three steps. First, it is proved that the theorem holds true if f is assumed to be twice continuously differentiable and strictly convex. Second, it is shown that the first step combined with an approximation argument yields the result for all strictly convex functions. Finally, another approximation argument is used to obtain the theorem in full generality.

Step 1. Observe that strict convexity of f implies strict convexity of F; that is, for any $x, y \in \mathbf{R}^n$ such that $x \neq y$, and $\lambda \in (0, 1)$ one has $F(\lambda x + (1-\lambda) y) < \lambda F(x) + (1-\lambda) F(y)$. By differentiability of f,

$$\nabla F(y) = -\int_{S^{n-1}} f'(\gamma(x) - \langle y, x \rangle)\, x \, d\sigma(x).$$

Let us recall that for any differentiable convex function $k: \mathbf{R}^n \to \mathbf{R}$ we have $\operatorname{Min} k = \{ y: \nabla k(y) = 0 \}$. Hence it remains to prove that there exists an $y \in A$ for which $\nabla F(y) = 0$. Define an auxiliary function $I(y, x)$ by the formula

$$I(y, x) = \tilde{\gamma}(x) - \langle y, x \rangle,$$

where $\tilde{\gamma}$ is the extension of γ ensured by Lemma 2.1. Since for each $y \in \mathbf{R}^n$ the function $x \mapsto f(I(y, x))$ is Lipschitz continuous on B^n, it follows from Proposition 3.1 that

$$-\nabla F(y) = \int_{B^n} f''(I(y, x))(\nabla \tilde{\gamma}(x) - y)\, dx. \tag{$**$}$$

For $y \in \mathbf{R}^n$, define

$$L(y) = \int_{B^n} f''(I(y, x))\, dx.$$

Since f is convex, L is nonnegative. Suppose now that $L(y) = 0$ for some $y \in \mathbf{R}^n$. By strict convexity of f, there exists a constant c such that for every $x \in B^n$

$$c = I(y, x) = \tilde{\gamma}(x) - \langle y, x \rangle.$$

Consequently, $y = \nabla \tilde{\gamma}(x)$ for almost all $x \in B^n$, and $\nabla F(y) = 0$. Moreover, from Proposition 2.1 it follows that $y \in A$. Hence we may further assume that the mapping L has only positive values. In such a case, $(**)$ can be expressed in the following manner

$$-\frac{\nabla F(y)}{L(y)} + y = \int_{B^n} \nabla \tilde{\gamma}(x) \frac{f''(I(y, x))}{L(y)}\, dx. \tag{$\dagger$}$$

Let $T(y)$ denote the left hand side of (†). Clearly, T is a continuous map from $\mathbf{R}^n$ into $\mathbf{R}^n$. Since the function $x \mapsto f'(I(y, x))/L(y)$ is a density of a probability measure, and $\nabla\tilde{y}$ maps B^n into A, the right hand side of (†) reads that T maps $\mathbf{R}^n$ into A. Now it is clear that Brouwer's theorem can be applied in order to draw the conclusion that T has a fixed point. This fixed point is an element of A and, at the same time, an argument for which the gradient ∇F equals 0.

Step 2. Now, we drop the hypothesis that f is differentiable and assume only strict convexity of f. As is well known, there exists an infinitely differentiable function ρ which satisfies the following three conditions:

(a) $\rho(s) \geqslant 0$, whenever $s \in \mathbf{R}$;

(b) $\int_{-\infty}^{+\infty} \rho(t)\, dt = 1$;

(c) $\operatorname{supp} \rho = \{t : \rho(t) \neq 0\} \subset [-1, 1]$.

It is a standard observation that every function ρ_r, $r > 0$, defined by the formula

$$\rho_r(s) = (1/r)\, \rho(s/r)$$

fulfils the first two conditions and that $\operatorname{supp} \rho_r \subset [-r, r]$. Let f_r denotes the *convolution* of f and ρ_r, that is,

$$f_r(s) = \int_{-\infty}^{+\infty} f(s-t)\, \rho_r(t)\, dt.$$

Let us recall that any such f_r is infinitely differentiable and that the net f_r, $r > 0$, converges uniformly to f on compact sets as $r \to 0$. Furthermore, it is obvious that all f_r are strictly convex. Let us replace f by f_r and F by F_r in (∗). Convergence of f_r implies that $F_r \to F$ uniformly on compact sets. Let ε be a positive number large enough to ensure $A \subset \varepsilon B^n$. Clearly, $F_r \to F$ uniformly on εB^n. By Step 1, $\min F_r$ belongs to A for each r. Subsequently, by Proposition 2.2, the restriction of F to εB^n has its minimizer belonging to A. Since ε is arbitrary, one deduces immediately that this minimizer is actually a minimizer of F.

Step 3. If f is not strictly convex then for every positive c one can define the function f_c,

$$f_c(s) = f(s) + cs^2,$$

It is obvious that all f_c are strictly convex, and that $f_c \to f$ uniformly on compact sets as $c \to 0$. Define F_c assuming in (∗) $F = F_c$ and $f = f_c$. As we have already proved that $\min F_c \in A$, it follows from Proposition 2.2, in much the same way as in Step 2, that for all sufficiently large ε, $\operatorname{Min} F_{|\varepsilon B} \cap A \neq \emptyset$. Hence $\operatorname{Min} F \cap A \neq \emptyset$. ▮

4. Centres

Let C and D be two elements of $\mathscr{K}^n$. Recall that

$$H_\infty(C, D) = \inf\{\varepsilon\colon D \subset C + \varepsilon B^n,\ C \subset D + \varepsilon B^n\}.$$

is the *Hausdorff distance* between C and D. It can also be written by the use of support functions

$$H_\infty(C, D) = \sup\{|\bar{h}_C(x) - \bar{h}_D(x)| : x \in S^{n-1}\} = |\bar{h}_C - \bar{h}_D|_\infty$$

The L^p metrics, $1 \leqslant p < \infty$, on $\mathscr{K}^n$ are defined as follows

$$H_p(C, D) = \left(\int_{S^{n-1}} |\bar{h}_C(x) - \bar{h}_D(x))|^p \, d\tilde{\sigma}(x) \right)^{1/p},$$

where $\tilde{\sigma}$ is the normalized Lebesgue measure on S^{n-1}. It is known (see [17]) that all the metrics H_p, $1 \leqslant p \leqslant \infty$, define the same topology on $\mathscr{K}^n$. We observe that the Hölder inequality implies

$$H_p(C, D) \leqslant H_q(C, D), \tag{††}$$

whenever $p \leqslant q$. Moreover, elementary calculation shows that

$$\lim_{p \to \infty} H_p(C, D) = H_\infty(C, D).$$

Let us fix some $A \in \mathscr{K}^n$ and define a mapping $G(y) = H_p(A, \{y\})$, where p is greater than one. Obviously, G has a unique minimizer. Recall that this minimizer is denoted by $m_p(A)$. It is evident that the function $F(y) = (G(y))^p$ has the same minimizer as G. Observe that F satisfies the formula $(*)$ of Theorem 3.2. provided that $f(s) = (1/\kappa)\, |s|^p$, where κ equals the area of the unit sphere, and $\gamma = \bar{h}_A$. Thus we have proved that $m_p(A)$ is an element of A. In fact, we can obtain even more

THEOREM 4.1. *Given $A \in \mathscr{K}^n$, $t \in \mathbf{R}$ and $p \in (1, \infty]$. Let $G_{p,t}$ denote the function*

$$G_{p,t}(y) = \begin{cases} H_p(A + tB^n, \{y\}), & \text{for } t \geqslant 0 \\ H_p(A, y + tB^n), & \text{for } t < 0. \end{cases}$$

There exists a unique element $m_{p,t}(A) \in \mathbf{R}^n$ *which minimizes* $G_{p,t}$; *this element belongs to* A.

Proof. First we consider the case $p \in (1, \infty)$. Let us define $F(y) = (G_{p,t}(y))^p$. Similarly as above, one observes that F satisfies $(*)$ if it is assumed that $f(s) = (1/\kappa)\ |s+t|^p$, and $\gamma = \bar{h}_A$. The uniqueness follows immediately from the strict convexity of f. The case $p = \infty$ is a simple consequence of the uniqueness of the Chebyshev center and the identity

$$m_\infty = m_{\infty,t} \tag{$\ddagger$}$$

which holds for all reals and can be easily derived from the definition of the Chebyshev center. ∎

THEOREM 4.2. *For every* $A \in \mathscr{K}^n$ *and every* $t \in \mathbf{R}^n$ *the mapping* $(1, \infty] \ni p \mapsto m_{p,t}(A)$ *is continuous. In particular,*

$$\lim_{p \to \infty} m_{p,t}(A) = m_\infty(A).$$

Proof. Let us fix $p \in (1, \infty]$. It is clear that the functions $G_{p,t}$ converge pointwise to $G_{p_0,t}$ as $p \to p_0$. Furthermore, we infer from (††) that these functions converge monotonically. By a well known theorem of Dini on monotone convergence of functions on compact sets, $G_{p,t}$ converges uniformly to $G_{p_0,t}$ on A. Now, the conclusion follows from Proposition 2.2, the preceding theorem, and the identity (‡) ∎

THEOREM 4.3. *For every* $A \in \mathscr{K}^n$ *and every* $p \in (1, \infty)$,

$$\lim_{t \to \pm\infty} m_{p,t}(A) = m_2(A)$$

Proof. To simplify the notation, let us write y_t instead of $m_{p,t}(A)$. Let us define the functions F, f and γ as in the proof of Theorem 4.1. It is clear that $\tilde{\gamma} = h_A$ and that $I(y, x) = h_A(x) - \langle y, x \rangle$ where I is as defined in the proof of Theorem 3.2. Elementary calculation shows that f is twice continuously differentiable except, possibly, at $s = -t$. As $y_t \in A$, it follows that there exists $\delta \geqslant 0$ such that $0 \leqslant I(y_t, x) \leqslant \delta$, whenever $t \in \mathbf{R}$ and $x \in B^n$. Thus the expression $f''(I(y_t, x))$ makes sens for all sufficiently large $|t|$ and $x \in B^n$. This observation enables us to employ the identity (†). Since y_t is a minimizer of F, one has $\nabla F(y_t) = 0$. By (†), we obtain

$$y_t = \int_{B^n} \nabla h_A(x)\, \frac{f''(I(y_t, x))}{L(y_t)}\, dx.$$

Now, notice that the mappings $(u, v) \mapsto f''(I(y_t, u))/f''(I(y_t, v))$ converge uniformly on $B^n \times B^n$ to the constant function 1 as $|t| \to \infty$. Consequently, the mappings $x \mapsto f''(I(y_t, x))/L(y_t)$ converge uniformly on B^n to the constant function $1/v$, where v denotes the volume of the unit ball. Thus we have

$$\lim_{t \to \infty} y_t = (1/v) \int_{B^n} \nabla h_A(x)\, dx.$$

Obviously, the right hand side is the Steiner point of the set A (compare: [16, §2] and [11, §6]). ∎

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